Probability theory - Class test 2

10 April, 2010, 10:00-12:00

<u>*Maximum marks is 50*</u>

Problem 1. [5 marks+5 marks] Let X_n be a sequence of random variables and let c_n denote a sequence of real numbers (non-random).

(1) Show that the sequence X_n is tight if and only if $\mathbf{P}(X_n > c_n) \to 0$ whenever $c_n \to \infty$.

(2) Show that the sequence X_n is uniformly integrable if and only if $\mathbf{E}\left[|X_n|\mathbf{1}_{|X_n|>c_n}\right] \to 0$ whenever $c_n \to \infty$.

Problem 2. [2 marks+2 marks+3 marks+3 marks] Let $X_1, X_2, \ldots$ be independent random variables. $S_n = X_1 + \ldots + X_n$ as usual. Which of the following events must necessarily have probability zero or one? Justify or give counterexample in each case. (a) $X_n > 1$ i.o. (b) $S_n > 1$ i.o. (c) $S_n - \lfloor S_n \rfloor > \frac{1}{2}$ i.o. (d) $\arcsin(S_n)$ converges. (For definiteness, let $\arcsin(x)$ take values in $[-\pi/2, \pi/2]$).

Problem 3. [10 marks] Let X_n, X be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$, show that some subsequence $X_{n_k} \xrightarrow{a.s} X$.

(2) If every subsequence of X_n has a further subsequence that converges almost surely to X, show that $X_n \xrightarrow{P} X$.

Problem 4. [5 marks+5 marks] Let $X_1, X_2, \ldots$ be i.i.d random variables with $\mathbf{E}[X_1] = 0$.

(1) Assume that $\mathbf{E}[X_1^2] < \infty$ and show that $\frac{X_1+2X_2+\ldots+nX_n}{1+2+\ldots+n} \xrightarrow{P} 0$.

(2) Assume that $\mathbf{E}[X_1^4] < \infty$ and show that $\frac{X_1+2X_2+\ldots+nX_n}{1+2+\ldots+n} \xrightarrow{a.s} 0$.

Problem 5. [10 marks] Let X_n be i.i.d random variables. Assume that $\mathbf{E}\left[\log_+ |X_1|\right] < \infty$, where $\log_+ x := \max\{\log x, 0\}$. Show that the power series $f(z) = \sum_n X_n z^n$ has radius of convergence equal to 1, a.s.

Problem 6. [10 marks] Let $X_1, X_2, \ldots$ be i.i.d $N(0,1)$ random variables. Let $a_1, a_2, \ldots$ be fixed (non-random) numbers. Show that $\sum_n a_n X_n$ converges a.s. if and only if $\sum_n a_n^2 < \infty$.